Econ 7010 - Midterm - 2026

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The exam is 75 minutes long, and has 77 marks (approx. 1 mark per minute). You may not use any outside materials, only writing implements. Write your answers in the booklet provided.

Short Answer - 7 marks each

1. The Cobb-Douglas model is: $Y = \beta_1 K^{\beta_2} L^{\beta_3} \varepsilon$. Can the parameters (the β_j) be estimated by LS? Why or why not?

   Assumption A.1 says the model must be linear in the parameters, and this model appears to be non-linear in the parameters, and so is unestimable by LS. However, taking the log of both sides transforms the model to:

   $$\log Y = \log \beta_1 + \beta_2 \log K + \beta_3 \log L + \log \varepsilon$$

   By using the logs of the variables, the model can be estimated.

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2. Prove that the residuals sum to zero when the model contains an intercept.

   By the FOC for the minimization of the sum-of-squared residuals (also called the "normal equations"):

   $$\begin{aligned} X'X\boldsymbol{b} &= X'\boldsymbol{y} \\ X'(X\boldsymbol{b} - \boldsymbol{y}) &= \mathbf{0} \\ X'\boldsymbol{e} &= \mathbf{0} \end{aligned}$$

   That is, the variables in X are orthogonal to the residuals $\boldsymbol{e}$. If the model contains an intercept, then one of the rows (e.g. the first) of X' is a row of 1s, and so $\sum e_i = 0$.

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3. Briefly describe the Frisch-Waugh-Lovell (FWL) theorem.

   The FWL theorem says that, in the partitioned model:

   $$\boldsymbol{y} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$$

   the LS estimates $\boldsymbol{b}_1$ for $\boldsymbol{\beta}_1$ may be obtained by estimating the model:

   $$\boldsymbol{y}^\star = X_1^\star\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}$$

where $\boldsymbol{y}^\star$ are the residuals from a regression of $\boldsymbol{y}$ on X_2, and $X_1^\star$ are the residuals from a regression of X_1 on X_2.

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4. Use FWL to explain why a model that uses centered data does not need to include an intercept. Do you think the residuals will still sum to zero (even though there's no intercept)?

Centering data (subtracting the mean of the variable from each value) is equivalent to regressing the variable on a column of 1s (an intercept), and taking the residuals. If all variables have been centered, then all variables are the residuals from a regression on an intercept. The FWL theorem says that the intercept can be dropped from the model without changing the estimated $\boldsymbol{\beta}$.

The residuals from the main model with centered data will still sum to zero, since there is effectively an intercept in the model according to FWL.

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5. Derive the variance-covariance matrix of the LS estimator $\boldsymbol{b}$. What assumption is important? What does this matrix have to do with the efficiency of LS? Why is this matrix important for hypothesis testing?

$$\begin{aligned} V(\boldsymbol{b}) &= V(\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{\epsilon} \\ &= (X'X)^{-1}X'V(\boldsymbol{\epsilon})X(X'X)^{-1} \quad \text{by A.5: non-random } X \\ &= (X'X)^{-1}X'\sigma^2 IX(X'X)^{-1} \quad \text{by A.4 (this is the important one)} \\ &= \sigma^2(X'X)^{-1} \end{aligned}$$

Gauss-Markov theorem proves that this is minimum variance under the standard assumptions, thus LS is efficient.

The diagonal elements of this matrix are used in the t-test, for example.

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6. Use the M or P matrices to show that changing the units of the variables (e.g. measuring in thousands of dollars instead of dollars) cannot change the model's R^2.

$$R^2 = \frac{\hat{y}'M_i\hat{y}}{y'M_iy} = 1 - \frac{e'e}{y'M_iy} \tag{1}$$

So, if changing the units of X doesn't change $\hat{\boldsymbol{y}}$ or $\boldsymbol{e}$, R^2 remains unchanged (similar logic for $\bar{R}^2$.

I'll show that $\hat{\boldsymbol{y}}$ doesn't change. Let A be a square $k \times k$ symmetric matrix that affects the change of units in X via XA, then:

$$\begin{aligned} P_{XA} &= XA(A'X'XA)^{-1}A'X' \\ &= XAA^{-1}(X'X)^{-1}A'^{-1}A'X' \quad \text{since A is square and symmetric} \\ &= X(X'X)^{-1}X' \end{aligned} \tag{2}$$

So, $P_{XA}\boldsymbol{y} = P_X\boldsymbol{y} = \hat{\boldsymbol{y}}$ and R^2 remains unchanged.

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7. Suppose that, very unrealistically, you magically knew the variance of the error term σ^2. Why do you think the z-test would be better to use than the t-test?

The z-test would be better because, instead of "wasting" information to estimate σ^2 using s^2, you could use σ^2 directly. This would result in a more powerful test.

Long Answer - 7 marks each part

9. (a) Suppose that you estimate the model

$$\boldsymbol{y} = X_1\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}$$

Prove that LS is unbiased, stating the important assumption that you need to make.

$$\begin{aligned} \boldsymbol{b}_1 &= (X_1'X_1)^{-1}X_1'(X_1\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}) \\ &= \boldsymbol{\beta}_1 + (X_1'X_1)^{-1}X_1'\boldsymbol{\epsilon} \\ E[\boldsymbol{b}_1] &= \boldsymbol{\beta}_1 + \mathbf{0} \quad \text{by A.5} \end{aligned}$$

(b) Suppose that the true model is actually:

$$\boldsymbol{y} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$$

but that you estimate the model from part (a) without X_2. Show that, in general, $\boldsymbol{b}_1$ is biased.

$$\begin{aligned} \boldsymbol{b}_1 &= (X_1'X_1)^{-1}X_1'(X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}) \\ &= \boldsymbol{\beta}_1 + (X_1'X_1)^{-1}X_1'X_2\boldsymbol{\beta}_2 + (X_1'X_1)^{-1}X_1'\boldsymbol{\varepsilon} \\ E[\boldsymbol{b}_1] &= \boldsymbol{\beta}_1 + (X_1'X_1)^{-1}X_1'X_2\boldsymbol{\beta}_2 + \mathbf{0} \end{aligned}$$

(c) Under what circumstances is $\boldsymbol{b}_1$ unbiased?

For the $(X_1'X_1)^{-1}X_1'X_2\boldsymbol{\beta}_2$ term to be 0 either (i) $X_1'X_2 = 0$ (they must be uncorrelated) or (ii) $\boldsymbol{\beta}_2 = \mathbf{0}$ (the true effect of X_2 on $\boldsymbol{y}$ must be 0).

(d) Now, suppose that the true model is $\boldsymbol{y} = X_1\boldsymbol{\beta}_1 + \boldsymbol{u}$, but that the model that you estimate is $\boldsymbol{y} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$. Show that, in this case, LS is unbiased.

$$\begin{aligned} \boldsymbol{b}_1 &= (X_1'M_2X_1)^{-1}X_1'M_2(X_1\boldsymbol{\beta}_1 + \boldsymbol{u}) \\ &= \boldsymbol{\beta}_1 + (X_1'M_2X_1)^{-1}X_1'M_2\boldsymbol{u} \\ E[\boldsymbol{b}_1] &= \boldsymbol{\beta}_1 + \mathbf{0} \quad \text{by A.5} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{b}_2 &= (X_2' M_1 X_2)^{-1} X_2' M_1 (X_1 \boldsymbol{\beta}_1 + \boldsymbol{u}) \\ &= 0 + (X_2' M_1 X_2)^{-1} X_2' M_1 \boldsymbol{u} \quad \text{since } M_1 X_1 = 0 \\ E[\boldsymbol{b}_2] &= \mathbf{0} \quad \text{by A.5} \end{aligned}$$

Since the true $\boldsymbol{\beta}_2 = \mathbf{0}$, $\boldsymbol{b}_2$ is unbiased.

END.

Econ 7010 Midterm Formula Sheet

standard regression model:	$y = X\beta + \epsilon$
LS estimator:	$b = (X'X)^{-1}X'y$
residual vector:	$e = (y - \hat{y}) = y - Xb$
estimator of error variance:	$s^2 = (e'e)/(n-k)$
covariance matrix for a random vector x:	$V(x) = E[(x - E(x))(x - E(x))']$
covariance matrix for errors:	$V(\epsilon) = \sigma^2 I_n$
"residual maker" matrix:	$M_X = I_n - X(X'X)^{-1}X'$
"fitted values maker" matrix:	$P_X = X(X'X)^{-1}X'$
deviations about means matrix:	$M_i = I_n - \frac{1}{n}ii'$
partitioned LS estimator:	$b_1 = (X_1' M_2 X_1)^{-1} X_1' M_2 y$
R-square:	$R^2 = \dfrac{\hat{y}' M_i \hat{y}}{y' M_i y} = 1 - \dfrac{e'e}{y' M_i y}$
adjusted R-square:	$\bar{R}^2 = 1 - \dfrac{e'e/(n-k)}{y' M_i y/(n-1)}$
t-statistic:	$t_j = \dfrac{b_j - \beta_j}{s.e.(b_j)} \sim t_{n-k}$