Econ 7010 - Midterm - Fall 2022

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The exam is 70 minutes long, and has 66 marks (approximately 1 mark per minute). You may not use any outside materials, only writing implements. Write your answers in the booklet provided. Each question part is worth 6 marks.

Short Answer - 3 sentences or 3 lines of math maximum

1. Show how $M_{\boldsymbol{\imath}}\boldsymbol{y}$ transforms $\boldsymbol{y}$ into deviations-from-means, where $\boldsymbol{\imath}$ is a column of 1s.

We need to show that $M_{\boldsymbol{\imath}}\boldsymbol{y}$ matrix is equal to $\boldsymbol{y} - \boldsymbol{\imath}\bar{y}$.

$$\begin{aligned} M_{\boldsymbol{\imath}} &= \left(I - \boldsymbol{\imath}\left(\boldsymbol{\imath}'\boldsymbol{\imath}\right)^{-1}\boldsymbol{\imath}'\right)\boldsymbol{y} \\ &= \boldsymbol{y} - \boldsymbol{\imath}\,(n)^{-1}\,\boldsymbol{\imath}'\boldsymbol{y} \\ &= \boldsymbol{y} - \boldsymbol{\imath}\left(\frac{1}{n}\right)\sum y_i = \boldsymbol{y} - \boldsymbol{\imath}\bar{y} \end{aligned}$$

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2. Will the R^2 or $\bar{R}^2$ change if the X variables are measured in different units? Explain.

$$R^2 = 1 - \frac{\boldsymbol{e}'\boldsymbol{e}}{\boldsymbol{y}'M_{\boldsymbol{\imath}}\boldsymbol{y}} \text{ and } \bar{R}^2 = 1 - \frac{\boldsymbol{e}'\boldsymbol{e}/(n-k)}{\boldsymbol{y}'M_{\boldsymbol{\imath}}\boldsymbol{y}/(n-1)}$$

A change in units of X amounts to a linear transformation of X. n, k, and $\boldsymbol{y}$ can not possibly change due to any transformations of the X variables. And since $\boldsymbol{e}$ is invariant to linear transformations of X, so is $\boldsymbol{e}'\boldsymbol{e}$ invariant, and so is R^2 and $\bar{R}^2$.

Intuition: $\bar{R}^2$ is a measure of the fit or "performance" of the model, it should not change if we were to measure X in different units or define dummy variables in alternate ways.

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3. Set up the derivation of the LS estimator.

$\boldsymbol{b}$ is chosen so as to minimize the sum of squared residuals:

$$\text{Min}_{(\boldsymbol{b})}\sum_{i=1}^{n} e_i^2$$

This is a minimization problem. Taking first derivatives, setting equal to zero, and solving, yields the least squares estimator.

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4. Explain why the t-test statistic does not follow a Normal distribution.

The t-statistic is:

$$t_i = \frac{(b_i - \beta_{i,0})}{\sqrt{s^2 \left[(X'X)^{-1}\right]_{ii}}}$$

While b_i is a Normally distributed variable, s^2 is Chi-square. The ratio of a Normal and Chi-square is not Normal.

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5. Prove that s^2 is mean-square consistent.

For a mean-square consistent estimator, bias and variance go to 0 an $n \to \infty$. s^2 is an unbiased estimator, and the variance of s^2 contains an n in the denominator (var $\left(s^2\right) = 2\sigma^4/(n-k)$), so as $n \to \infty$, var $\left(s^2\right) \to \infty$.

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6. Explain the difficulty in comparing the asymptotic distributions of any two consistent estimators.

Any two consistent estimators both have asymptotic distributions that are indistinguishable (a "spike" located at the true parameter value). In both cases, the variance is vanishing. We must instead scale the estimators (usually by $\sqrt{n}$), in order to compare their asymptotic variances.

Long Answer

7. Consider a model with just an intercept, and a single $\boldsymbol{x}$ variable:

$$\boldsymbol{y} = \beta_1 \boldsymbol{\imath} + \beta_2 \boldsymbol{x} + \boldsymbol{\epsilon}$$

Suppose that both $\boldsymbol{y}$ and $\boldsymbol{x}$ have been "centred".

a) What do you think the estimated value of β_1 will be?

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b_1 will equal 0.

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b) Prove your answer in part (a).

There are at least (iii) ways to prove.

i. In a model with only an intercept and single x variable, the formula for the intercept is:

$$b_1 = \bar{\boldsymbol{y}} - \bar{\boldsymbol{x}} b_2$$

Since $\bar{\boldsymbol{y}} = \bar{\boldsymbol{x}} = 0$, $b_1 = 0$.

ii. From the FOC for the derivation of the LS estimator, $X'\boldsymbol{e} = \mathbf{0}$. When an intercept is included in the model, this implies that the LS regression line will pass through the sample means of the data. Since the $\boldsymbol{y}$ and $\boldsymbol{x}$ data both have mean 0, the estimated regression line must pass through $(0, 0)$, and so the intercept must be 0.

iii. The partitioned formula for b_1 is:

$$b_1 = \left(\boldsymbol{\imath}' M_x \boldsymbol{\imath}\right)^{-1} \boldsymbol{\imath}' M_x \boldsymbol{y}$$

and

$$\boldsymbol{\imath}' M_x = \boldsymbol{\imath}' M_x' = (M_x \boldsymbol{\imath})' = \boldsymbol{\imath}'$$

$M_x \boldsymbol{\imath} = \boldsymbol{\imath}$ because regressing a column of 1s on $\boldsymbol{x}$ (no intercept in the model) would give an LS line that is flat and passes through the origin, and so all residuals would equal 1. Finally, since $\boldsymbol{y}$ has mean 0, $\boldsymbol{\imath}'\boldsymbol{y} = \sum y_i = 0$, and so $b_1 = 0$.

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8. Consider the model:

$$\boldsymbol{y} = X_1 \boldsymbol{\beta}_1 + X_2 \boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$$

suppose that all of the usual assumptions are satisfied, except that the error term is correlated to X_2: $\mathrm{E}[\boldsymbol{\epsilon}] = X_2 \boldsymbol{\gamma}$.

a) Show that LS is biased, in general (show that $\mathrm{E}[\boldsymbol{b}] \neq \boldsymbol{\beta}$).

$$\begin{aligned}
\boldsymbol{b} &= \left(X'X\right)^{-1} X'\boldsymbol{y} \\
&= \left(X'X\right)^{-1} X' \left(X\boldsymbol{\beta} + \boldsymbol{\epsilon}\right) \\
&= \boldsymbol{\beta} + \left(X'X\right)^{-1} X'\boldsymbol{\epsilon} \\
\mathrm{E}\left[\boldsymbol{b}\right] &= \boldsymbol{\beta} + \left(X'X\right)^{-1} X' X_2 \boldsymbol{\gamma} \neq \boldsymbol{\beta}
\end{aligned}$$

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b) Show that only $\boldsymbol{b}_2$ is biased.

$$\begin{aligned}
\boldsymbol{b}_1 &= \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 \boldsymbol{y} \\
&= \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 \left(X_1 \boldsymbol{\beta}_1 + X_2 \boldsymbol{\beta}_2 + \boldsymbol{\epsilon}\right) \\
&= \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 X_1 \boldsymbol{\beta}_1 + \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 X_2 \boldsymbol{\beta}_2 + \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 \boldsymbol{\epsilon} \\
&= \boldsymbol{\beta}_1 + \mathbf{0} + \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 \boldsymbol{\epsilon} \\
\mathrm{E}\left[\boldsymbol{b}_1\right] &= \boldsymbol{\beta}_1 + \left(X_1' M_2 X_1\right)^{-1} X_1' M_2 X_2 \boldsymbol{\gamma} = \boldsymbol{\beta}_1
\end{aligned}$$

$\boldsymbol{b}_1$ is unbiased, so that only $\boldsymbol{b}_2$ is biased.

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9. Use the Frisch-Waugh-Lovell theorem to explain the problem with estimating a model that uses some variables that have been de-seasonalized or de-trended, and some variables that have not.

The FWL theorem says that the models:

$$\boldsymbol{y}^\star = X_1^\star \boldsymbol{\beta}_1 + \boldsymbol{u}$$

and

$$\boldsymbol{y} = X_1 \boldsymbol{\beta}_1 + X_2 \boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$$

yield identical estimators for $\boldsymbol{\beta}_1$, where $\boldsymbol{y}^\star$ and $X_1^\star$ are the residuals from regressions of $\boldsymbol{y}$ on X_2, and X_1 on X_2, respectively. All the variables in the model are regressed on X_2 before X_2 can be removed without altering the LS estimate $\boldsymbol{b}_1$.

De-trending or de-seasonalizing a variable is the same as regressing it on a time trend or on seasonal dummy variables, and taking the residuals. Unless all variables are de-trended or de-seasonalized in the same way, it is not the same as estimating a model that includes a time trend or seasonal dummies.

END.