1. White's test for heteroskedasticity may be implemented by regressing the squared residuals from OLS on the original X variables and their squares and cross-products, and determining if the $nR^2$ from this regression is large.

The null and alternative hypotheses in White's test are

$$H_0: var(\varepsilon_i) = \sigma^2 \quad \text{(homoskedasticity)}$$
$$H_A: var(\varepsilon_i) = \sigma_i^2 \quad \text{(heteroskedasticity)}$$

If the null hypothesis is rejected, there is no prescription as to the form of heteroskedasticity, and so the test is non-constructive. As the alternative hypothesis is very general, the idea is to try to approximate the unknown form of het. using X, squares of X, and cross-products. If variation in $e_i^2$ can be explained, we should reject $H_0$. That is, if $nR^2$ is too large we should reject ($nR^2 \sim \chi^2$).

2. The algorithm is:

$$\theta_{n+1} = \theta_n - H^{-1}(\theta_n) g(\theta_n)$$

$$f(\theta) = \theta^3 - 3\theta$$
$$g(\theta) = 3\theta^2 - 3$$
$$H(\theta) = 6\theta \quad ; \quad H^{-1}(\theta) = 1/6\theta \quad , \quad \text{so:}$$

$$\theta_{n+1} = \theta_n - \frac{3\theta_n^2 - 3}{6\theta_n}$$

We need to choose a starting value, $\theta_0$, for the algorithm. Note that if you pick $\theta_0 = 0$ then you're going to have a bad time (at $f(0)$ there is an inflection point).

Try $\theta_0 = -2$.

$$\theta_1 = -2 - (3(4) - 3) / 6(-2) = -1.25$$
$$\theta_2 = -1.25 - [3(1.5625) - 3] / 6(-1.25) = -1.025$$
$$\theta_3 = -1.000305$$

We see this is converging to $-1$. However, since $H(-1) < 0$, $\theta = -1$ actually solves for the max. of $f(\theta)$, and we are looking for the min. If instead we start at $\theta_0 = 2$, for example, we'll see that $\theta_n$ converges to $1$, which is the location of the max of $f(\theta)$.

3. As per the notation in class, let:

$$V(\varepsilon) = \sigma^2 \Omega$$

The GLS estimator is $\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$. Since $\Omega$ is unknown, it must be estimated by some consistent estimator, $\hat{\Omega}$. Replacing $\Omega^{-1}$ with $\hat{\Omega}^{-1}$ in the GLS formula gives the FGLS estimator:

$$\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}y$$

Since OLS is still unbiased and consistent even when $V(\varepsilon) = \sigma^2\Omega$, so are the OLS residuals unbiased and consistent estimators for $\varepsilon$. Hence, we can use $\mathbf{e}$ to estimate $\sigma^2\Omega$, even if we do not know the form of het. That is, we can estimate $\Omega$ by:

$$\hat{\Omega} = \begin{bmatrix} e_1^2 & & 0 \\ & e_2^2 & \\ 0 & & \ddots & \\ & & & e_n^2 \end{bmatrix}$$

4. Since $\lim_{n\to\infty} Bias(s^2) = \lim_{n\to\infty} E(s^2) - \sigma^2 = 0$, and

$\lim_{n\to\infty} var(s^2) = \lim_{n\to\infty} \frac{2\sigma^4}{n-K} = 0$, $s^2$ is a Mean Square consistent estimator.

Similarly, $\lim_{n\to\infty} Bias(\hat{\sigma}^2) = \lim_{n\to\infty} \frac{\sigma^2(n-k)}{n} - \sigma^2 = 0$, and

$\lim_{n\to\infty} var(\hat{\sigma}^2) = 0$, $\hat{\sigma}^2$ is also mean-square consistent.

5. The RLS is unbiased, consistent and efficient, if the restrictions are true. If the restrictions are false it is ~~un~~ biased and inconsistent, but RLS will still have smaller variance than OLS. To estimate RLS, OLS can be applied to the restricted model.

6. Since $\text{plim}\left(s^2\left(\frac{X'X}{n}\right)^{-1}\right) = \text{plim}(s^2) \cdot \text{plim}\left(\frac{X'X}{n}\right)^{-1}$

$= \sigma^2 Q^{-1}$, we have that $\text{plim}\left(s^2\left(\frac{X'X}{n}\right)^{-1}\right) = V(\sqrt{n}\, b)$.

Hence, $n\,\text{plim}\left(s^2(X'X)^{-1}\right) = n V(b)$ and $\text{plim}\left(s^2(X'X)^{-1}\right)$ should equal $V(b)$.

7. a) If, in general, $V(\varepsilon) = \sigma^2\Omega$, the variance of the OLS estimator is:

$$V(b) = V[(X'X)^{-1}X'\varepsilon] = (X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}$$

$$= (X'X)^{-1}X'\sigma^2\Omega X(X'X)^{-1}$$

In this case, $V(\varepsilon) = \sigma^2 \operatorname{diag}(X_2)$, so $\Omega = \operatorname{diag}(X_2)$, and:

$$V(b) = (X'X)^{-1}X'\sigma^2 \operatorname{diag}(X_2)X(X'X)^{-1}$$

b) OLS will still be unbiased and consistent, but it will be inefficient. However, $s^2(X'X)^{-1}$ will be an inconsistent estimator for $V(b)$, so hypothesis testing will be invalid.

c) $$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = (X'\operatorname{diag}(X_2)^{-1}X)^{-1}X'\operatorname{diag}(X_2)^{-1}y$$

d) For $\hat{\beta}$ to be the GLS estimator, it must be the case that $P'P = \Omega^{-1}$.

e) $$P = \begin{bmatrix} 1/\sqrt{X_{12}} & & & 0 \\ & 1/\sqrt{X_{22}} & & \\ & & \ddots & \\ 0 & & & 1/\sqrt{X_{n2}} \end{bmatrix}$$

8. a) (i) $X = Z\gamma + \epsilon$

$$\hat{\gamma} = (Z'Z)^{-1}Z'X$$

$$\hat{X} = Z(Z'Z)^{-1}Z'X$$

(ii) $y = \hat{X}\beta + v$

$$\hat{\beta} = (\hat{X}'\hat{X})^{-1}\hat{X}'y$$

$$= (X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

$$= (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

Which is the generalized (or over-identified) IV estimator.

b) If X and Z have the same dimensions, then we can write the generalized estimator as:

$$\hat{\beta}_{IV} = (Z'X)^{-1}\underbrace{(Z'Z)\underbrace{(X'Z)^{-1}(X'Z)}_{I}(Z'Z)^{-1}}_{I}Z'y$$

$$= (Z'X)^{-1}Z'y$$

c) We can use the Hausmann test. The null hypothesis is that OLS is consistent. If $H_0$ is true, then both IV and OLS are consistent, and $b_{OLS} - \hat{\beta}_{IV} \approx 0$. The test statistic is based on this difference, with a large difference suggesting rejection of the null hypothesis.