Econ 7010 - Midterm 1 2016 - Answer Key

1. B 2. C 3. A 4. C 5. C

6. From the F.O.C. for solving for b we have:

$$X'Xb - X'y = 0$$
$$X'(Xb - y) = 0$$
$$X'e = 0$$

If the model includes an intercept, then one of the columns of X (for example the 1^{st} column) is a column of 'ones'. So,

$$X'e = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & & \cdots & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \Sigma e_i \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

That is, the first element of $X'e$ is Σe_i, which must be equal to 0 by the F.O.C.

7. $V(b) = V\left[(X'X)^{-1}X'y\right]$

$$= V\left[(X'X)^{-1}X'(X\beta + \varepsilon)\right]$$

$$= V\left[(X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon\right]$$

$$= V\left[\beta + (X'X)^{-1}X'\varepsilon\right]$$

$$= \cancelto{0}{V(\beta)} + V\left[(X'X)^{-1}X'\varepsilon\right]$$

By A.5, X is non-random, and:

$$V(b) = \cancelto{0}{V(\beta)} + (X'X)^{-1}X'V(\varepsilon)X(X'X)^{-1}$$

By A.4, $V(\varepsilon) = \sigma^2 I$, and:

$$V(b) = (X'X)^{-1}X'\sigma^2 I X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}X'X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}$$

Implicitly, we are also using A.1 and A.2

8.a) $E(\tilde{\beta}_1) = E\left[\frac{1}{n}\sum\left(\frac{y_i - \bar{y}}{x_i - \bar{x}}\right)\right]$

$$= E\left[\frac{1}{n}\sum\left(\frac{\beta_0 + \beta_1 x_i + \varepsilon_i - \beta_0 - \beta_1 \bar{x} - \bar{\varepsilon}}{x_i - \bar{x}}\right)\right]$$

$$= \frac{1}{n} E\left[\sum\left(\frac{\beta_1 x_i + \varepsilon_i - \beta_1 \bar{x} - \bar{\varepsilon}}{x_i - \bar{x}}\right)\right]$$

$$= \frac{1}{n}\sum\left(\frac{\beta_1 x_i + E(\varepsilon_i) - \beta_1 \bar{x} - E(\bar{\varepsilon})}{x_i - \bar{x}}\right)$$

$$= \frac{1}{n}\sum\left(\frac{\beta_1 x_i - \beta_1 \bar{x}}{x_i - \bar{x}}\right) = \frac{1}{n}\sum \beta_1 = \beta_1$$

b) $\tilde{\beta}_1$ is linear since it can be written as a weighted sum of the y data,

$$\tilde{\beta}_1 = \sum c_i(y_i - \bar{y}),$$

where $c_i = \frac{1}{n} \frac{1}{x_i - \bar{x}}$

c) $$\text{var}(\tilde{\beta}_1) = \text{var}\left[\frac{1}{n} \sum \left(\frac{y_i - \bar{y}}{x_i - \bar{x}}\right)\right]$$

$$= \frac{1}{n^2(x_i - \bar{x})^2} \text{var} \sum (y_i - \bar{y})$$

$$= \frac{1}{n^2(x_i - \bar{x})^2} \sum [\text{var}(y_i) + \text{var}(\bar{y})]$$

$$= \frac{n\sigma^2 + \sigma^2}{n^2(x_i - \bar{x})^2}$$

While we could compare the variance above to that of $\text{var}(b_1)$, we would find that $\text{var}(\tilde{\beta}_1) > \text{var}(b_1)$, due to the Gauss-Markov theorem. The inequality is strict since $\tilde{\beta}_1$ is not the OLS estimator.

9. The population model is:

$$y = X\beta + \varepsilon. \quad (1)$$

With the addition of the dummy variable, the model becomes:

$$y = X_1\beta_1 + X_2\beta_2 + \varepsilon, \quad (2)$$

where $X_1\beta_1 = X\beta$ from (1), and where $X_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The OLS estimator for β_1 in (2) is:

$$b_1 = (X_1'M_2X_1)^{-1}X_1'M_2y,$$

where $M_2 = I - X_2(X_2'X_2)^{-1}X_2'$

$$= I - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (1)^{-1} [1 0 \cdots 0]$$

$$= I - \begin{bmatrix} 1 & & \emptyset \\ & 0 & \\ \emptyset & & \ddots & 0 \end{bmatrix} = \begin{bmatrix} 0 & & \emptyset \\ & 1 & \\ \emptyset & & \ddots & 1 \end{bmatrix}$$

Note that $X_1'M_2X_1 = X_1'X_1 - X_1'\begin{bmatrix} I & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} X_1$.

$X_1'M_2y$ is computed similarly. This just drops the first observation.

Alternatively, write the estimator for β_1 as:

$$b_1 = (X_1'M_2'M_2X_1)^{-1} X_1'M_2'M_2y$$

$$= \left(\begin{bmatrix} 0 & x_{2k} & & x_{nk} \\ \vdots & \vdots & \cdots & \\ 0 & x_{21} & & x_{n1} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ x_{21} & \cdots & x_{2n} \\ & \vdots & \\ x_{n1} & & x_{nK} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & x_{2k} & & x_{nk} \\ \vdots & \vdots & \cdots & \\ 0 & x_{21} & & x_{n1} \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Note that $M_2X_1 = \begin{bmatrix} 0 & \cdots & 0 \\ x_{21} & \cdots & x_{2k} \\ & \vdots & \\ x_{n1} & \cdot & x_{nk} \end{bmatrix}$

$b_1 = \sum_{i=2}^{n} c_i y_i$, where c_i is a non-random constant determined by the X data.