Department of Economics **University of Manitoba**

ECON 7010: Econometrics I
FINAL EXAM, Dec. 13th, 2016

Instructor: **Ryan Godwin**
Instructions: **Put all answers in the booklet provided.**
Time Allowed: **3 hours.**
Number of Pages: **7**

There are a total of 100 marks.

PART A: Short answer. ***Answer 5 out of 8 questions.*** *Each question is worth 6 marks.*

1.)

2.) Consider the simple linear regression model, where all of the usual assumptions are satisfied. Two potential estimators for σ^2 are:

$$s^2 = \frac{e'e}{n-k} \ ,$$

and:

$$\hat{\sigma}^2 = \frac{e'e}{n} \ .$$

By taking the expected values of these estimators we find that s^2 is unbiased and $\hat{\sigma}^2$ is biased:

$$E[s^2] = \sigma^2 \ ,$$

$$E[\hat{\sigma}^2] = \frac{n-k}{n}\sigma^2 \ .$$

Assuming that ε is Normally distributed, it can be shown that:

$$var(s^2) = \frac{2\sigma^4}{n-k} \ ,$$

$$var(\hat{\sigma}^2) = \frac{2\sigma^4}{n} \ .$$

Are s^2 and $\hat{\sigma}^2$ consistent estimators for σ^2, in the case of Normally distributed errors? Prove.

3.) Briefly describe the properties of a Maximum Likelihood Estimator.

4.) Consider the population model:

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \varepsilon,$$

where all of the usual assumptions are satisfied, except that ε is **not** Normal. Suppose you want to test the null hypothesis:

$$H_0\colon \beta_1 + \beta_2 = 1 \; and \; \beta_5 = 0.$$

i) What are the R and q matrices for this test?

ii) What test would you use and what is the distribution of the test statistic?

5.) In the basic linear model estimated by OLS, where the model includes an intercept, prove that the estimated population regression "line" (regression hyperplane) passes through the sample means of the data.

6.) Suppose that there is no closed-form solution for the estimator in your model, hence, you must use numerical methods (such as the Newton-Raphson algorithm) to solve for the value of the estimator. Explain one situation (graphically if you like) where the algorithm may fail to converge on a solution.

7.) Explain why the coefficient of determination, R^2, cannot decrease when a regressor is added to the model.

8.) Explain what a p-value is and why a low p-value suggests the null hypothesis should be rejected.

PART B: Long answer. **You must answer Question 1. It is a difficult question, worth only 10 marks. You might want to leave this until the end of the exam.**

1.) Suppose that we have a standard linear multiple regression model, with *k* regressors:

$$y = X\beta + \varepsilon, \tag{1.1}$$

where *all of the usual assumptions are satisfied*. In addition, we have some further information in the form of J *uncertain* restrictions on the parameters:

$$R\beta = q + v, \tag{1.2}$$

where R is $(J \times k)$, with $rank(R) = J \ (< k)$; q is $(J \times 1)$; the elements of both R and q are known; and v is a random error that reflects the uncertainty associated with the restrictions. You may assume that $v \sim N(0, V)$, and ε and v are uncorrelated.

a) Re-arrange equation (1.2) and then "stack up" this equation below equation (1.1) in a form that enables you to estimate the coefficient vector, β.

b) Show that the estimator for β, obtained by applying OLS to the "stacked" model, is

$$\hat{\beta} = [X'X + R'R]^{-1}[X'y + R'q]$$

c) Prove that this estimator is unbiased, under the stated assumptions.

d) Derive the expression for the covariance matrix of this estimator.

e) Suppose that $V = \sigma^2 I$. Prove that in this case the covariance matrix of $\hat{\beta}$ simplifies to become $\sigma^2[X'X + R'R]^{-1}$.

f) For the situation in part (e), explain how you would prove that $\hat{\beta}$ is more efficient than the OLS estimator, when applied just to the model in (1.1).

g) For the situation in part (e), explain how you would construct a test statistic and actually test if one of the elements of the β vector takes a particular value, (say) β_0.

PART C: Long answer with choice. **Answer 3 of the 5 following questions.** 20 marks each. Each part has equal weight unless otherwise stated.

2.)

3.) Suppose that we are working with cross-section data. We want to estimate a linear regression model:

$$y_j = \beta_1 + \beta_2 x_{2j} + \cdots + \beta_k x_{kj} + \varepsilon_j \quad ; \quad j = 1, 2, \ldots, n. \qquad (3.1)$$

This model satisfies *all* of the usual assumptions. The only problem is that we are not provided with individual data for the *n* values of each of the variables. Instead, the data have been gathered by conducting a survey across m groups of people, and then recording the group average values. There are different numbers if people (n_i) in each group, and we have this information as well. So, the data that are available are:

$$\bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_j \;\; ; \; \bar{x}_{2i} = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{2j} \;\; ; \; \ldots \; ; \; \bar{x}_{ki} = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{kj} \;\; ; \; and \; n_i \; ; \; i = 1, 2, \ldots, m.$$

This means that the model we have to estimate is actually:

$$\bar{y}_i = \beta_1 + \beta_2 \bar{x}_{2i} + \cdots + \beta_k \bar{x}_{ki} + \bar{\varepsilon}_i \; ; \; i = 1, 2, \ldots, m. \qquad (3.2)$$

a) Show that the variance of $\bar{\varepsilon}_i$ in equation in equation (3.2) is (σ^2/n_i), where σ^2 is the variance of each ε_j in equation (3.1). What are the other properties of the error term in equation (3.2)?

b) Briefly explain how you would estimate equation (3.2) by Weighted Least Squares (WLS). Why would it be preferable to use this estimator, rather than applying OLS to equation (3.2)?

c) Suppose that *k* = 2, and we have the following data:

i	n_i	$\bar{y}_i$	$\bar{x}_{2i}$
1	16	3	2
2	4	2	3
3	9	5	1
4	4	1	4

Show that the WLS estimates of β_1 and β_2 are approximately 5.99 and -1.34 respectively.

d) Now suppose that you do not have information on the group size, n_i. In this case, briefly explain how you could implement Feasible Generalized Least Squares (FGLS).

4.)

5.) Suppose that we want to estimate the following k – regressor model by Instrumental Variables (IV) estimation, using a matrix of ‘L’ (> k) instruments, Z:

$$y = X\beta + \varepsilon$$

a) List the three conditions that we require the matrix of instruments to satisfy.

6 marks

b) Assuming that these conditions are satisfied, prove that the IV estimator of β is (weakly) consistent.

7 marks

c) Now suppose that the true data-generating process is

$$y = X\beta + W\gamma + \varepsilon$$

where W is a matrix of observations for p additional random regressors, such that

$$plim\left(\frac{1}{n}Z'W\right) \neq 0,$$

and $L > (p + k)$. Prove that the IV estimator in (b) is now inconsistent.

7 marks

6.) A discrete random variable, $= 0, 1, 2, 3, \ldots$, is said to have a Poisson distribution with parameter $\lambda > 0$, if the probability mass function of y is given by:

$$p(y \mid \lambda) = \frac{\lambda^y}{e^\lambda y!}$$

The mean and variance of y are both equal to λ.

a) Assuming independence, write down the likelihood function $L(\lambda \mid y_1, y_2, \ldots, y_n)$

b) Derive the maximum likelihood estimator for λ in this model.

c) Prove that this estimator is unbiased.

d) Explain how you could alter the probability mass function (and likelihood function) so that you would have a regression model (that is, a model where the variable y is a function of x variables).

e) Explain the intuition behind the Likelihood Ratio test *or* the Score (Lagrange Multiplier) test.

END.