Econ 7010 - Midterm 1 Answer Key

Part A - Multiple Choice

1. D   2. B   3. C   4. A

5.a) $V(b) = E[(b - E(b))(b - E(b))']$

Using A.3, the OLS estimator is unbiased, and $E(b) = \beta$.

Now, $b - \beta = (X'X)^{-1}X'y - \beta$

$$= (X'X)^{-1}X'(X\beta + \varepsilon) - \beta$$

$$= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon - \beta$$

$$= (X'X)^{-1}X'\varepsilon$$

So, $V(b) = E[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}]$

By A.5, the X matrix is non-random, so:

$V(b) = (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1}$

By A.4, $E(\varepsilon\varepsilon') = \sigma^2 I_n$. So:

$V(b) = (X'X)^{-1}X'\sigma^2 X(X'X)^{-1}$

$$= \sigma^2(X'X)^{-1}.$$

We are implicitly assuming that the OLS estimator exists, so we also need A.2.

b) If the model includes an intercept, then :

$$R^2 = 1 - \frac{SSE}{SST}.$$

For SSE = 0, $e'e = 0$, and so $e = 0$.

This is only true if the regression includes an intercept. In this case, we have a "perfect fit."

7. If $E(\hat{\theta}) = \frac{n-c}{n} \theta$, then :

$$E\left(\frac{n}{n-c} \hat{\theta}\right) = \theta.$$

Let $\tilde{\theta} = \frac{n}{n-c} \hat{\theta}$, then $\tilde{\theta}$ is unbiased for θ.

If we replace "c" with "K", "$\hat{\theta}$" with "$\hat{\sigma}^2$", then $\tilde{\theta} = s^2$, our estimator for the variance of the error term.

b) By the Gauss-Markhov theorem, we know that if $\tilde{\beta}$ is a linear and unbiased estimator, then:

$$V(\tilde{\beta}) - V(b) = \text{a p.s.d. matrix}$$

That is, the variance of $\tilde{\beta}$ cannot be less than the variance of $\mathbb{b}$.

6.a) $R^2 = \frac{SSR}{SST}$. If the model includes an intercept, then:

$$R^2 = 1 - \frac{SSE}{SST}, \text{ where } SSE = e'e.$$

Now, $\mathbb{b}$ is derived by: $\min_{\mathbb{b}} e'e$.

Consider two optimization problems:

$$\min_{\mathbb{b}_1} (y - X_1\mathbb{b}_1)(y - X_1\mathbb{b}_1)' \qquad (1)$$

$$\min_{b_1, \mathbb{b}_2} (y - X_1 b_1 - X_2 b_2)(y - X_1\mathbb{b}_1 - X_2 b_2)' \qquad (2)$$

Optimization problem (1) is just optimization problem (2), subject to the constraint: $\mathbb{b}_2 = 0$. Hence, the sum of squared residuals from (2) cannot be higher than from (1), and the R^2 from (2) must be larger.

8.) Algebraically:

$$MX = (I - X(X'X)^{-1}X')X = X - X(X'X)^{-1}X'X$$

$$= X - X = 0$$

Intuitively:

M is a "residual maker." When pre-multiplied by a vector, it creates the residuals from an OLS regression of that vector, on X. Hence, MX is creating the residuals from a regression of X on X. In this case, we have "perfect fit", and the residuals are 0.

9.) a) $e = y - \hat{y} = y - X\mathbf{b} = y - X(X'X)^{-1}X'y$

$$= \cancel{y}(I - X(X'X)^{-1}X')\,y$$

$$= (I - X(X'X)^{-1}X')(X\beta + \varepsilon)$$

$$= X\beta + \varepsilon - X(X'X)^{-1}X'X\beta - X(X'X)^{-1}X'\varepsilon$$

$$= X\beta + \varepsilon - X\beta - X(X'X)^{-1}X'\varepsilon = \varepsilon - X(X'X)^{-1}X'\varepsilon$$

So, $E(e) = 0 - 0 = 0.$

Here, we have used A3: $E(\varepsilon) = 0$, and A2: full rank.

b.) From the normal equations:

$$X'Xb = X'y$$

$$X'Xb - X'y = 0$$

$$X'(Xb - y) = 0$$

$$X'e = 0.$$

We only need to assume linearity.