Final Exam 2014 – Answer Key

1.) The following steps may be used to implement White's test for heteroskedasticity:

1) Estimate the model by OLS. Obtain the residual vector, e.
2) Perform an auxillary regression, where $e'e$ is regressed on all 'X' variables, and the squared and cross-products of all 'X' variables.
3) If the R^2 is high enough (from the auxillary regression), then the null of homoskedasticity is rejected. The test statistic is nR^2, and is χ^2 distributed.

This test is not constructive, as rejection does not indicate the form of heteroskedasticity.

2.) A spurious regression occurs when one variable, which is a random walk, is regressed on another variable which is also a random walk, for example. In a spurious regression, the two variables are independent, but the R^2 from such a regression tends to 1 as n goes to infinity. A variable, y_t, which follows a random walk has the following property:

$$y_t = y_{t-1} + \varepsilon_t$$

3) If the error term is AR(1), then:

$$\varepsilon_t = \rho \varepsilon_{t-1} + u_t,$$

where $|\rho| < 1$, and u_t satisfies all of the classical assumptions about an error term. In this case, the assumption about spherical disturbances is violated, since the error term exhibits autocorrelation, and OLS is inefficient. FGLS may be used instead, however, a consistent estimator for the Ω matrix is required.

In this case, Ω is a function only of the parameter, ρ. So, a consistent estimator may be found by estimating ρ consistently. This may be done by running OLS, obtaining the residuals, e_t, and then estimating:

$$e_t = r e_{t-1} + \varepsilon_t$$

The OLS estimator for r may then be used in place of ρ in the Ω matrix, and FGLS may be implemented.

4) Heteroskedasticity can affect OLS in two important ways. It causes the usual estimator for $V(b)$ to be inconsistent. That is, $s^2(X'X)^{-1}$ is based on the wrong formula. Hypothesis testing involving b will be invalid. A remedy to this problem is to base an estimator on the correct formula:

$$V(b) = (X'X)^{-1}X'\sigma^2\Omega X(X'X)^{-1}$$

For example, White's Het. Robust estimator for $V(b)$ is based on the OLS residuals.

Another problem which arises is the inefficiency of OLS. In the presence of heteroskedasticity, GLS or FGLS are efficient estimators.

5) An AR(1) process is:

$$\varepsilon_t = \rho\,\varepsilon_{t-1} + u_t \quad ; \quad |\rho| < 1 .$$

An MA(∞) process is:

$$\varepsilon_t = u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots$$

If we iterate the AR(1) process back one period:

$$\varepsilon_{t-1} = \rho\,\varepsilon_{t-2} + u_{t-1},$$

and substitute into the original equation:

$$\varepsilon_t = \rho^2 \varepsilon_{t-2} + \rho u_{t-1} + u_t$$

If we iterate back two periods, we can eliminate ε_{t-2} from above:

$$\varepsilon_t = \rho^3 \varepsilon_{t-3} + \rho^2 u_{t-2} + \rho u_{t-1} + u_t$$

Continuing in this fashion we get:

$$\varepsilon_t = u_t + \rho u_{t-1} + \rho^2 u_{t-2} + \rho^3 u_{t-3} + \dots ,$$

which is an MA(∞) process.

6) Since (from the formula sheet):

$$e'e = y'M_0 y\,(1-R^2_u)\,,$$
$$e^{*\prime}e^* = y'M_0 y\,(1-R^2_R)\,,$$

we can rewrite the F-stat formula so it is in terms of R^2_U and R^2_R.

$$F = \frac{(e^{*\prime}e^* - e'e)\,/\,J}{e'e\,/\,(n-k)} = \frac{y'M_0 y\,(R^2_u - R^2_R)\,/\,J}{y'M_0 y\,(1-R^2_u)\,/\,(n-k)}$$

$$= \frac{(R^2_u - R^2_R)\,/J}{(1-R^2_u)\,/\,(n-k)}$$

So, $F = \dfrac{(0.5-0.4)\,/\,2}{(1-0.5)\,/\,100} = 10.$

7) $b_* = b - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(Rb-q)$

$E(b_*) = \beta - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\beta - q)$

If the null hypothesis is correct, then $R\beta = q$ and

$E(b_*) = \beta$.

8) A p-value obtained from a test statistic is the probability of obtaining a value for the test statistic which is more extreme than the one just calculated, given the null hypothesis is true.

11.) a) We require:

i) $\text{plim}\left(\frac{Z'X}{n}\right) = Q_{ZX}$, a p.s.d. matrix. That is, the instruments are correlated with the regressors.

ii) $\text{plim}\left(\frac{Z'\varepsilon}{n}\right) = 0$, i.e. the instruments are (asymptotically) uncorrelated with the error term.

b) $b_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + \varepsilon)$

$$= \beta + (Z'X)^{-1}Z'\varepsilon = \beta + \left(\frac{Z'X}{n}\right)^{-1}\left(\frac{Z'\varepsilon}{n}\right)$$

Using the conditions above, and Slutsky's theorem:

$$\text{plim}(b_{IV}) = \beta + Q_{ZX}^{-1} \cdot 0 = \beta$$

c) $b_{IV} = (Z'X)^{-1}Z'(X\beta + W\gamma + \varepsilon)$

$$= \beta + (Z'X)^{-1}Z'W\gamma + (Z'X)^{-1}Z'\varepsilon = \beta + \left(\frac{Z'X}{n}\right)^{-1}\left(\frac{Z'W}{n}\right)\gamma + \left(\frac{Z'X}{n}\right)^{-1}\left(\frac{Z'\varepsilon}{n}\right)$$

Let $\text{plim}\left(\frac{1}{n}Z'W\right) = Q_{ZW} \neq 0$, say. Then:

$$\text{plim}(b_{IV}) = \beta + Q_{ZX}^{-1}Q_{ZW}\gamma + Q_{ZX} \cdot 0 = \beta + Q_{ZX}^{-1}Q_{ZW}\gamma.$$

12) a) $var(\bar{\varepsilon}_i) = var\left(\frac{1}{n_i}\sum_{j=1}^{n_i} \varepsilon_j\right) = \frac{1}{n_i^2} var(\Sigma \varepsilon_j)$

$= \frac{1}{n_i^2} \Sigma\, var(\varepsilon_j)$ (by independence of the ε_j's)

$= \frac{1}{n_i^2} n_i \sigma^2 = \frac{\sigma^2}{n_i}$

Also: $E(\bar{\varepsilon}_i) = E\left(\frac{1}{n_i}\Sigma\varepsilon_j\right) = 0$, since $E(\varepsilon_j) = 0$,

and $\bar{\varepsilon}_i$ will be normally distributed since it is a linear function of normally distributed variables.

b) $V(\bar{\varepsilon}) = \sigma^2\Omega = \sigma^2 \begin{bmatrix} \frac{1}{n_1} & & & 0 \\ & \frac{1}{n_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{n_m} \end{bmatrix}$

So, $\Omega^{-1} = \begin{bmatrix} n_1 & & \\ & \ddots & \\ & & n_m \end{bmatrix}$,

and $P = \begin{bmatrix} \sqrt{n_1} & & \\ & \ddots & \\ & & \sqrt{n_m} \end{bmatrix}$,

so that $P'P = \Omega^{-1}$.

We can apply GLS to (2) by applying OLS to a transformed model, where the transformed model is:

$$\sqrt{n_i}\,\bar{y}_i = \sqrt{n_i}\,\beta_1 + \beta_2\sqrt{n_i}\,\bar{X}_{2i} + \dots + \beta_K\sqrt{n_i}\,\bar{X}_{Ki} + \sqrt{n_i}\,\bar{\varepsilon}_i$$

This would be preferable, as GLS is efficient, whereas OLS is not.

13) a) $L(\lambda \mid y) = \prod_{i=1}^{n} \lambda e^{-y_i\lambda} = \lambda e^{-\sum_{i=1}^{n} y_i\lambda}$

$$\log L = \log \lambda - \sum_{i=1}^{n} y_i\lambda$$

b) $\dfrac{\partial \log L}{\partial \lambda} = \dfrac{1}{\lambda} - \sum y_i = 0$ (FOC)

$$\tilde{\lambda} = 1/\sum y_i$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{-1}{\lambda^2} = (-) \text{ for } \lambda > 0.$$

So, $\tilde{\lambda}$ maximizes the log-likelihood.

c)

14. a) Suppose that the fitted model is:

$$y = X_1\beta_1 + \varepsilon,$$

but that the true model is:

$$y = X_1\beta_1 + X_2\beta_2 + u.$$

The OLS estimator from the fitted model is:

$$b_1 = (X_1'X_1)^{-1}X_1'y.$$

Substitute the true model in for y:

$$b_1 = (X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2 + u)$$

$$= (X_1'X_1)^{-1}X_1'X_2\beta_2 + (X_1'X_1)^{-1}X_1'u + \beta_1$$

$$E(b_1) = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2.$$

So, OLS is biased, in general. It will be unbiased, however, if $X_1'X_2 = 0$ (X_1 and X_2 are uncorrelated), or if $\beta_2 = 0$.

b) Fitted model: $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$

True model: $y = X_1\beta_1 + u$

$b_1 = (X_1' M_2 X_1)^{-1} X_1' M_2 y$, where $M_2 = I - X_2(X_2'X_2)^{-1}X_2'$.

$b_1 = (X_1' M_2 X_1)^{-1} X_1' M_2 X_1 \beta_1 + (X_1' M_2 X_1)^{-1} X_1' M_2 u$

$= \beta_1 + (X_1' M_2 X_1)^{-1} X_1' M_2 u$

$E(b_1) = \beta_1 + 0$,

So b_1 is unbiased.

Similarly, we will find $E(b_2) = \beta_2$. Since $\beta_2 = 0$ in the true model, b_2 is also unbiased.