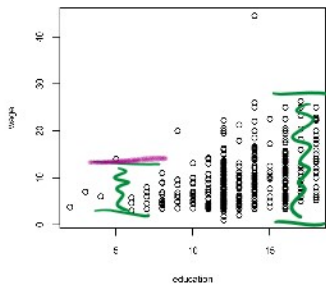




### ECON 3040 - Heteroskedasticity

Ryan T. Godwin  
University of Manitoba

Figure: Possible heteroskedasticity in the CPS data. The variance in wage may be increasing as education increases. The reasoning is that individuals who have not completed highschool (or university) are precluded from many high-paying jobs (doctors, lawyers, etc.). However, having many years of education does not preclude individuals from low-paying jobs. The spread in wages is higher for highly educated individuals.



$var(wage) \neq \sigma^2$

1. Define heterosked./homosked.

↳ sentence/formula

2. What are the implications of het.?

↳ s.e.'s are wrong (inconsistent) ⇒ invalid hyp. testing

↳ LS is inefficient (bonus)

3. How to test?

↳ White's test ( $H_0/H_a$ )

↳ use residuals in a 2<sup>nd</sup> stage model

↳ look at  $R^2$

4. How to fix?

↳ use "robust" se's (diff. formula for var(b))

↳ doesn't change "b"

↳ changes se's / t-stats / p-val's / CI's

### We have been assuming homoskedasticity

The estimators that we have used so far have good statistical properties provided that the following assumptions hold:

1. The population model is linear in the  $\beta$ s.
2. There is no perfect multicollinearity between the  $X$  variables.
3. The random error term,  $\epsilon$ , has mean zero.
4.  $\epsilon$  is identically and independently distributed. → same  $\sigma^2$  for everyone
5.  $\epsilon$  and  $X$  are independent.
6.  $\epsilon$  is Normally distributed.

These ensure LS is unbiased, efficient, and consistent, and that hypothesis testing is valid. A violation of one or more of these assumptions might lead us to estimators beyond LS.

We will consider that assumption 4 is violated in a particular way. Specifically, we consider what happens where the error term,  $\epsilon$ , is not identically distributed.

same dispersion  
Homoskedasticity "equal variance"

If assumption A4 is satisfied, then  $\epsilon$  is identically distributed. This means that all of the  $\epsilon_i$  have the same variance. That is, all of the random effects that determine  $Y_i$ , outside of  $X_i$ , have the same dispersion. The term homoskedasticity (same dispersion) refers to this

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Stated mathematically, homoskedasticity means:

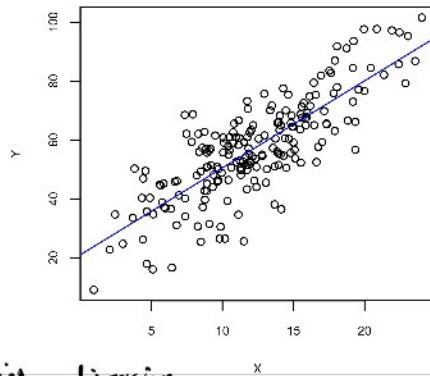
$$\text{Var}[\epsilon_i | X_i] = \sigma^2, \forall i \quad \text{homosked.}$$

The variance of  $\epsilon$  is constant, even conditional on knowing the value of  $X$ .

Homoskedasticity means that the squared vertical distance of each data point from the (population or estimated) line is, on average, the same. The values of the  $X$  variables do not influence this distance (the variance of the random unobservable effects are not determined by any of the values of  $X$ ). See figure 2.

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Figure: Homoskedasticity. The average squared vertical distance from the data points to the OLS estimated line is the same, regardless of the value of  $X$ .



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*differing dispersion*

Heteroskedasticity "unequal variances"

Heteroskedasticity refers to the situation where the variance of the error term  $\epsilon$  is not equal for all observations. The term heteroskedasticity means *differing dispersion*. Mathematically:

$$\text{Var}[\epsilon_i | X_i] \neq \sigma^2, \forall i$$

or

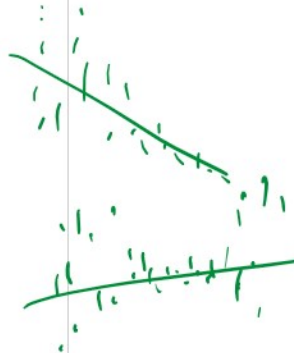
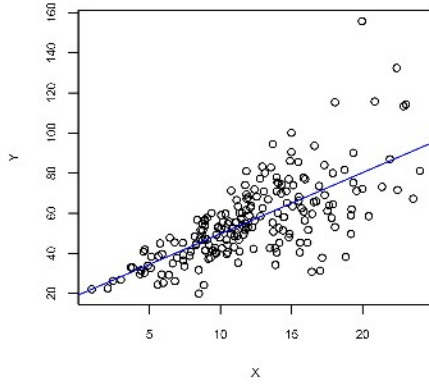
$$\text{Var}[\epsilon_i | X_i] = \sigma_i^2$$

Each observation can have its own variance, and the value of  $X$  may influence this variance.

Heteroskedasticity means that the squared vertical distance of each data point from the estimated regression line is not the same on average, and may be influenced by one or more of the  $X$  variables.

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Figure: Heteroskedasticity. The squared vertical distance of a data point from the OLS estimated line is influenced by X.



### The implications of heteroskedasticity

Heteroskedasticity is a violation of A.4, since each  $\epsilon_i$  is not identically distributed. Heteroskedasticity has two main implications for the estimation procedures we have been using in this book:

1. The OLS estimator is no longer efficient. *→ not a big problem*
2. The estimated standard errors are inconsistent.

The inefficiency of OLS is arguably a smaller problem than the inconsistency of the variance estimator. The second issue means that the estimated standard errors in our regression output are wrong, leading to the incorrect t-statistics and confidence intervals.

Hypothesis testing, in general, is invalid. The problem arises because the formula that is the basis for estimating the standard errors in OLS:

$$\text{Var}(b_1) = \frac{\sigma^2}{\sum X_i^2 - \frac{(\sum X_i)^2}{n}}$$

*assumes homosked. → replace w/  $S^2_{\epsilon}$*

is only correct under homoskedasticity.

*Fix: GLS weighted LS*  
*→ use low variance ppl more*  
*not in this course*

$$t = \frac{b_k - \beta_{k,0}}{\sqrt{\text{var}(b_k)}}$$

### Fixing heteroskedasticity - robust standard errors

To fix the more important problem of the inconsistency of the standard errors, the formula for  $\text{Var}(b_1)$  must be updated to take into account the possibility of heteroskedasticity.

Updating the formula to allow for heteroskedasticity in the estimation of the standard errors gives what is typically referred to as *robust standard errors*. In R, we will use the code:

```
1 install.packages("ltest")
2 library(ltest)
3 install.packages("sandwich")
4 library(sandwich)
```

*instead of  $s^2 = \frac{\sum \epsilon_i^2}{n-k}$*   
 *$X \epsilon_i X$*

to install and load a package that can estimate the robust standard errors, and then use

```
coefLust(my.lm.model, vcov = vcovHC(my.lm.model, "HC1"))
```

to estimate the correct standard errors and updated t-statistics and p-values, where `my.lm.model` is the least-squares regression that we have estimated using the `lm()` command.

### Testing for heteroskedasticity

There are several (approximately) equivalent tests for heteroskedasticity, but we'll focus on the most famous: White's<sup>1</sup> test. In White's test, the null hypothesis is that there is homoskedasticity, and the alternative is heteroskedasticity. That is:

$$H_0: \text{var}(\epsilon_i) = \sigma^2 \text{ homosked.}$$

$$H_A: \text{var}(\epsilon_i) = \sigma_i^2 \text{ heterosked.}$$

$$\text{var}(\epsilon_i) \neq \sigma^2 \text{ heterosked.}$$

*Residuals can be used to test:*  
*- heterosked.*  
*- normality*  
*can't be used for X and E are independent*

## Testing for heteroskedasticity

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$$\text{var}[\epsilon_i] \neq \sigma^2 \quad \text{heterosked.}$$

<sup>1</sup>White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica: Journal of the Econometric Society*, 817-838.

Residuals  
can be  
used to test:  
- heterosked.  
- normality  
can't be used  
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are independent

Take a simple population model with two regressors. Remember that the population model and the estimated model are (respectively):

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon \quad \text{true}$$

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \hat{\epsilon} \quad \text{estimated}$$

The residual  $\hat{\epsilon}$  is the counterpart to the unobservable error term  $\epsilon$ . Sometimes, we can use the residuals to test assumptions or properties of the error term. For example, we can look to see if the residuals are homoskedastic or heteroskedastic, in order to infer those properties about the error term. That is, if  $\hat{\epsilon}$  looks homoskedastic, we will conclude that so is  $\epsilon$ .

1<sup>st</sup> stage: estimate pop. model  $\rightarrow$  get  $e_i$ 's

2<sup>nd</sup> stage (auxiliary regression):  $e_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \alpha_3 x_{1i}^2 + \alpha_4 x_{2i}^2 + \alpha_5 x_{1i} x_{2i} + \epsilon_i$   
 $\rightarrow$  get  $R^2 \rightarrow$  if high  $\rightarrow$  reject

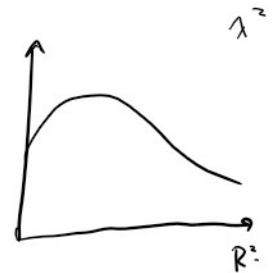
White's test:

$$H_0: \text{homosked.} (\text{var}(\epsilon_i) = \sigma^2)$$

$$H_A: \text{heterosked.} (\text{var}(\epsilon_i) \neq \sigma^2)$$

White's test tries to explain differences in the size of the squared residuals from a least-squares model by regressing them on the original  $x$  variables, and the squares and cross products of the  $x$ . If the  $R^2$  from this regression is high, then we conclude that there is some pattern to the size of the residuals, and reject the null hypothesis of homoskedasticity.

$$H_0: \text{var}(\epsilon_i) = \sigma^2 \quad \text{if true} \rightarrow R^2 = 0$$



To test for heteroskedasticity in the population model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

we would estimate it by LS, for example by using `lm(y ~ x1 + x2)`. We then get the squared residuals from this regression, and estimate the following equation by LS:

$$e^2 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3(x_1 \times x_2) + \beta_4 x_1^2 + \beta_5 x_2^2 + \varepsilon \quad (1)$$

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Equation 1 is looking for any approximate way to explain variation in the size of the squared residuals. If the estimated model from equation 1 fits well (in terms of the R-squared), then there is some explanation for the variance in the error term, and the error term is heteroskedastic. White's test statistic is the  $nR^2$  from this auxiliary regression, and the p-value for the test comes from the Chi-square distribution. As usual, if the p-value is small, we reject the null of homoskedasticity, in favour of heteroskedasticity.

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To test for heteroskedasticity in R, we need to install and load a package:

```
1 install.packages("skedastic")  
2 library(skedastic)
```

and then use:

```
1 write(my_lm_model, interactions = TRUE)
```

where `my_lm_model` is the model we have estimated by LS. If we find heteroskedasticity, then we need to use heteroskedastic robust standard errors (such as White's standard errors).

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## Heteroskedasticity in food expenditure data

Download a data set on food expenditure by country, in 2016:

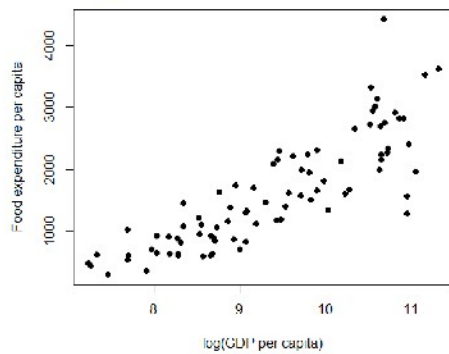
```
1 food <- read.csv("https://xtgodyn.com/data/foodexp.csv")
```

The variables are `foodexp` - food expenditure per capita (in US dollars), and `GDPpercap` - GDP per capita. There are 84 countries in the sample. Plot the data, taking the log of GDP per capita (see Figure 4):

```
1 plot(log(food$GDPpercap), food$foodexp, pch=16, xlab="log(  
  GDP per capita)",  
2 ylab="Food expenditure per capita")
```

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Figure: Food expenditure and log GDP per capita.



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## Estimate the population model

The following model for food expenditure:

$$\text{foodexp} = \beta_0 + \beta_1 \log(\text{GDPpercap}) + \epsilon$$

can be estimated in R using:

```
1 food_mod <- lm(foodexp ~ log(GDPpercap), data=food)  
2 summary(food_mod)
```

```
1 Coefficients:  
2 (Intercept) -4737.68 651.38 -10.60 <2e-16 ***  
3 log(GDPpercap) 677.40 47.81 14.17 <2e-16 ***  
4 ---  
5 Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
6  
7 Residual standard error: 479 on 82 degrees of freedom  
8 Multiple R-squared: 0.71, Adjusted R-squared: 0.7065  
9 F-statistic: 209.8 on 1 and 82 DF, p-value: < 2.2e-16
```

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## Test for heteroskedasticity

If heteroskedasticity is present in this data, then the standard errors, t-statistics, and p-values, are all wrong! Hypothesis testing, and any conclusions we draw, may be incorrect due to the heteroskedasticity. To test for heteroskedasticity, we can use White's test:

```
1 install.packages("skedastic")
2 library(skedastic)
3 white(food.mod)

1  statistic  p.value  parameter  method  alternative
2  <dbl>    <dbl>    <dbl>    <chr>    <chr>
3  11.6     0.00304    2        White's Test greater
```

The test statistic from the White test is 11.6, with an associated p-value of 0.00304. We reject the null hypothesis of homoskedasticity.

$hR^2$  → reject  $H_0$  of homosked.

$$e_i^2 = \beta_0 + \beta_1 \log(GDPpercap) + \beta_2 \log(GDPpercap)^2 - \epsilon_i$$

1/2/20

To see what the function `white()` is doing, we'll calculate the White test statistic and p-value "by hand":

```
1 food.resid.sq <- food.mod$residuals ^ 2
2 summary(lm(food.resid.sq ~ log(GDPpercap) + I(log(GDPpercap)
3   ^ 2), data=food))
```

```
1 Coefficients:
2             Estimate Std. Error t value Pr(>|t|)
3 (Intercept)    4703168    3680579   1.278   0.205
4 log(GDPpercap) -1121479     795218  -1.410   0.162
5 I(log(GDPpercap)^2)  67768    42508   1.593   0.115
6
7 Residual standard error: 444800 on 81 degrees of freedom
8 Multiple R-squared:  0.138, Adjusted R-squared:  0.1167
9 F-statistic: 6.485 on 2 and 81 DF,  p-value: 0.002442
```

The test statistic is  $nR^2 = 84 \times 0.138 = 11.6$  (same as from the `white()` command). The p-value can be found from:

```
1 1 - pchisq(84 * 0.138, 2)
0.003039689
```

which is the same from the `white()` command.

"robust standard errors"

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## White's heteroskedastic consistent standard errors

To recalculate the standard errors, t-statistics, and p-values, we can use the `coeftest()` function:

```
1 install.packages("lmtest")
2 library(lmtest)
3 install.packages("sandwich")
4 library(sandwich)
5 coeftest(food.mod, vcov = vcovHC(food.mod, "HC1"))

1 t test of coefficients:
2
3             Estimate Std. Error t value Pr(>|t|)
4 (Intercept)  -4737.680    476.616  -9.9423 9.705e-18 ***
5 log(GDPpercap)  677.399     64.068   10.5754 < 2.2e-16 ***
6 ---
7 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Notice that the estimated  $\beta$ s have not changed, but that the standard errors have changed, t-statistics, and p-values have changed.

updates se for heterosked.

3/7/20

Heteroskedastic errors have a pretty severe consequence: hypothesis testing may be invalid. The prevalence of heteroskedasticity in many economics data has led to the common practice of erring on the side of caution. Heteroskedastic robust standard errors are often used, if heteroskedasticity is suspected. Note that homoskedasticity is a special case of heteroskedasticity, so the downside of using the robust estimator when it is not needed, is small.

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