Ch. 7-1

Wednesday, February 26, 2025 10:30 AM

ch7-1

7 – Joint Hypothesis Tests

Now that we have multiple "X" variables, and multiple βs, our hypotheses might also involve more than one β.

- We shouldn't use t-tests
- We should use the F-test

The types of hypotheses we are now considering involve multiple coefficients (βs). For example:

q = 2

$$H_0 : \beta_1 = \beta_2 = 0$$
$$H_A : \beta_1 \neq 0 \text{ and/or } \beta_2 \neq 0$$ not H₀

and

q = 3

$$H_0 : \beta_1 = 1, \beta_2 = 2, \beta_4 = 5$$
$$H_A : \cancel{\beta_1 \neq 1 \text{ and/or } \beta_2 \neq 2 \text{ and/or } \beta_4 \neq 5}$$ not H₀

Note that the null hypothesis is wrong if *any* of the individual hypotheses about the βs are wrong. In the latter example, if $\beta_2 \neq 2$, then the whole thing is wrong. Hence the use of the "and/or" operator in H_A. It is common to omit all the "and/or" and simply write "not H_0" for the alternative hypothesis.

- A joint hypothesis specifies a value (imposes a restriction) for two or more coefficients (βs)
- Use q to denote the number of restrictions ($q = 2$ for 1st example, $q = 3$ for second example)

F-tests can be used for *model selection*. Which variables should we leave out of the model?

- If variables are insignificant ($H_0: \beta = 0$), we might want to drop them from the model
- Dropping a variable means we hypothesize its β is zero
- Dropping multiple variables at once means all of the associated βs are all zero

Example: CPS data again

hourly wage — years of educ

```
summary(lm(wage ~ education + gender + age + experience))

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)    -1.9574     6.8350  -0.286    0.775
education       1.3073     1.1201   1.167    0.244
genderfemale   -2.3442     0.3889  -6.028 3.12e-09 ***
age            -0.3675     1.1195  -0.328    0.743
experience      0.4811     1.1205   0.429    0.668
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.458 on 529 degrees of freedom
Multiple R-squared:  0.2533, Adjusted R-squared:  0.2477
F-statistic: 44.86 on 4 and 529 DF,  p-value: < 2.2e-16
```

$H_0: \beta_{educ} = 0$ ↳ fail to reject

$\boxed{H_0: \beta_{age} = 0}$

↳ $t = |-0.328| < 1.96 \Rightarrow$ fail to reject

↳ p-val $= 0.743 > 0.05 \Rightarrow$

Why drop a variable

1. makes model simpler
2. improves efficiency

The results of the above regression make me want to drop `age` and `experience`.

This corresponds to the (joint) hypothesis:

$$\boxed{H_0: \beta_3 = 0 \textit{ and } \beta_4 = 0}$$

H_A: either $\beta_3 \neq 0$ *or* $\beta_4 \neq 0$ *or both* — not H_0

Why would we want to drop variables?

1. simpler is better
2. reduces variance of estimators

We can't use *t*-tests

Idea (doesn't work): reject H_0 if **either** $|t_3|$ > 1.96 **and/or** $|t_4|$ > 1.96.

Review: type I error = Pr (reject H_0 | H_0 is true) = α

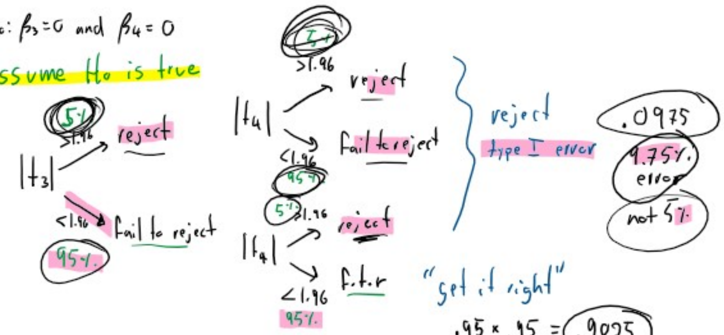
Exercise: Assuming that t_3 and t_4 are *independent*, show that the type I error for the above test is 9.75% (not 5%).

How would you correct this problem? (Bonferroni method – not used in practice)

↘ increase the 1.96

H_0: $\beta_3 = 0$ and $\beta_4 = 0$

Assume H_0 is true



$.95 \times .95 = .9025$

$1 - .9025 = .0975$

A bigger problem: t_3 and t_4 are likely *not* independent

In the model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \epsilon$$

correlated

- suppose that X_3 and X_4 are *not* independent (e.g. they are correlated)
- then the OLS estimators b_3 and b_4 will be correlated - the formula for b_3 (etc.) involves *all* of the "X" variables (remember OVB)
- then t_3 and t_4 will be correlated!

$$t_3 = \frac{b_3 - \beta_{3,0}}{s.e.(b_3)} \qquad t_4 = \frac{b_4 - \beta_{4,0}}{s.e.(b_4)}$$

correlated

correlated

$b_3 = f(y, X_1, X_2, X_3, X_4)$

$b_4 = f(y, X_1, X_2, X_3, X_4)$

Example

Suppose that X_3 and X_4 are positively correlated. Consider the null:

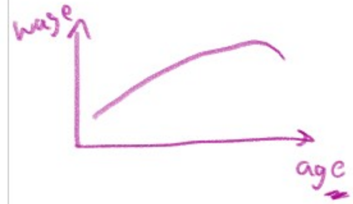
$$H_0: \beta_3 = 0 \text{ and } \beta_4 = 0$$

- if b_3 and b_4 are both positive (or negative), it's not that big of a deal
- if one is positive and the other negative, that's a big deal

CPS data again

```
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)     -1.9574     6.8350  -0.286    0.775
education        1.3073     1.1201   1.167    0.244
genderfemale    -2.3442     0.3889  -6.028 3.12e-09 ***
age             -0.3675     1.1195  -0.328    0.743  - insig.
experience       0.4811     1.1205   0.429    0.668  - insig.
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$\beta_{age} = \frac{\Delta wage}{\Delta age}$; all else equal



- do the signs of the coefficients make sense? ✓
- what is the sign of the correlation between `age` and `experience`? (+)
- according to the two individual t-tests, we fail to reject the null:

$$\boxed{H_0: \beta_3 = 0 \text{ and } \beta_4 = 0}$$

Let's try the F-test

I'm going to estimate two models:

- One model under the **alternative hypothesis** – we'll call the **unrestricted model** (the βs are allowed to be anything) — H_A (full model)
- One model under the **null hypothesis** – called the **restricted model.** — H_0 — I get this model by taking the null hypothesis to heart. That is, substitute in the values $\beta_3 = 0$ and $\beta_4 = 0$ into the full model

Unrestricted model (under H_A): $wage = \beta_0 + \beta_1 educ + \beta_2 gender + \beta_3 age + \beta_4 exper + \epsilon$

```
unrestricted <- lm(wage ~ education + gender
    + age + experience)
```

Restricted model (under H_0): $\beta_3 = 0$ and $\beta_4 = 0$

```
restricted <- lm(wage ~ education + gender)
```

$wage = \beta_0 + \beta_1 educ + \beta_2 gender + \epsilon$

F-test command:

```
anova(unrestricted, restricted)
```

Output (F-stat in blue, p-val in red):

OPPOSITE result

```
Analysis of Variance Table

Model 1: wage ~ education + gender + age + experience
Model 2: wage ~ education + gender
  Res.Df   RSS Df Sum of Sq      F    Pr(>F)
1    529 10511
2    531 11425 -2   -914.27 23.007 2.625e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

0.000000000002625 → reject Ho

Interpretation? (A big F-stat still means reject)

A formula for the F-test statistic

- The F-test takes into account the correlation between the estimators that are involved in the test
- Note that if the unrestricted model "fits" significantly better than the restricted model, we should reject the null. HA
- The difference in "fit" between the model under the null and the model under the alternative leads to a formulation of the F-test statistic, for testing joint hypotheses.

The RSS is a measure of fit:

min Σe_i^2 ↓ $b_0, b_1, \cancel{b_2}$

$$RSS = \sum_{i=1}^{n} e_i^2$$

$H_0: \beta_2 = 0$

where

$$e_i = Y_i - \hat{Y}_i$$

The F-test statistic may be written as:

bigger smaller

$$F = \frac{(RSS_{restricted} - RSS_{unrestricted})/q}{RSS_{unrestricted}/(n - k_{unrestricted} - 1)}$$

2

if big → reject

where q = # of restrictions, k = # of "X"s

Notice that if the restrictions are true (if the null is true), $RSS_{restricted} - RSS_{unrestricted}$ will be small, and we'll fail to reject.

$$\boxed{RSS = \frac{1 - R^2}{TSS}}$$

Another statistic which uses RSS is the R^2:

$$\boxed{R^2 = 1 - \frac{RSS}{TSS}}$$

solve for this
↳ plug into F-stat above

This gives us another formula for the F-test statistic:

$$F = \frac{(\overset{H_A}{\overset{\text{bigger}}{R^2_{unrestricted}}} - \overset{H_0}{\overset{\text{smaller}}{R^2_{restricted}}})/q}{(1 - R^2_{unrestricted})/(n - k_{unrestricted} - 1)}$$

where:

$R^2_{restricted}$ = the R^2 for the restricted regression

$R^2_{unrestricted}$ = the R^2 for the unrestricted regression

q = the number of restrictions under the null

$k_{unrestricted}$ = the number of regressors in the unrestricted regression.

The bigger the difference between the restricted and unrestricted R^2's – the greater the improvement in fit by adding the variables in question – the larger is the F statistic.

Testing you on the exam

- The F-test statistic can be obtained by comparing the R^2 in the restricted model (H_0 model) and the unrestricted model (H_A model).
- The decision to reject or not depends on whether the F-stat exceeds the (5%) critical value:

q	5% critical value
1	3.84
2	[redacted]
3	[redacted]
4	[redacted]
5	[redacted]

→ like 1.96 crit. value

- These values are only accurate if n is large (we'll always assume this)

Exercise

Test

$$\boxed{H_0: \beta_3 = 0 \text{ and } \beta_4 = 0}$$

in the model:

$$wage = \beta_0 + \beta_1 education + \beta_2 genderfemale + \beta_3 age + \beta_4 experience + \epsilon$$

H_A

H_A unrestricted

5

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)    -1.9574     6.8350  -0.286    0.775    
education       1.3073     1.1201   1.167    0.244    
genderfemale   -2.3442     0.3889  -6.028 3.12e-09 ***
age            -0.3675     1.1195  -0.328    0.743    
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F-statistic: 44.86 on 4 and 529 DF,  p-value: < 2.2e-16
```

H_0 restricted

3

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)   0.21783    1.03632   0.210    0.834    
education     0.75128    0.07682   9.779  < 2e-16 ***
genderfemale -2.12406    0.40283  -5.273 1.96e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.639 on 531 degrees of freedom
Multiple R-squared:  0.1884, Adjusted R-squared:  0.1853 
F-statistic: 61.62 on 2 and 531 DF,  p-value: < 2.2e-16
```

$$F = \frac{(R^2_U - R^2_R)/q}{(1-R^2_U)/(n-K_U-1)}$$

$$= \frac{(0.2533 - 0.1884)/2}{(1-0.2533)/(\underbrace{534-4-1}_{529})} \approx 23$$

compare to crit. value